(M_1, g) (M_2, h) $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $G = f_2^2 g + f_1^2 h$

$M_1 \times M_2$ f_1 f_2 M_1 M_2

1969 Bishop

[1] 2016

[2] 2018

[3] 2022

Levi-Civita Ricci [4]

[5-6]

[7] 1982 Michelsohn

[8] 1 0 [8] 2014

[9]

1985 Balas [10] Ricci

λ λ [10]

1987 Kobayashi [11]

(M, J, G) n J G M p

$$T_p^{\mathbb{C}} M = T_p^{1,0} M \oplus T_p^{0,1} M$$

J $\pm\sqrt{-1}.$ $z = (z^1, z^2, \cdots, z^n)$ $T_p^{1,0} M$ $(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^n})$

∇ G J

[3]

$$\Gamma^{\beta}_{\gamma\alpha} = G^{\bar{\delta}\beta}\frac{\partial G_{\alpha\bar{\delta}}}{\partial z^{\gamma}} \quad 1$$

[12]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$T^{\beta}_{\gamma\alpha} = \Gamma^{\beta}_{\gamma\alpha} - \Gamma^{\beta}_{\alpha\gamma} \quad 2$$

1 3 K [11]

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

$$K^{\beta}_{\alpha\gamma\bar{\delta}} = -\frac{\partial \Gamma^{\beta}_{\gamma\alpha}}{\partial \bar{z}^{\delta}} \quad 3$$

1 3 [11]

$$K^{\beta}_{\alpha} = G^{\bar{\delta}\gamma}K^{\beta}_{\alpha\gamma\bar{\delta}} \quad 4$$

[10] (M, J, G)

$$T^{\beta}_{\gamma\alpha} = 0 \quad 5$$

(M, J, G)

[8] (M, J, G) 1 0

$$\tau_{\gamma} = 0 \quad 6$$

(M, J, G)

$$\tau_{\gamma} = G^{\bar{\eta}\alpha}T_{\gamma\alpha\bar{\eta}} \quad 7$$

$$T_{\gamma\alpha\bar{\eta}} = G_{\beta\bar{\eta}}T^{\beta}_{\gamma\alpha} \quad 8$$

[11] (M, J, G)

$$K = \varphi I, \quad i.e., \quad K^{\beta}_{\alpha} = \varphi\delta^{\beta}_{\alpha} \quad 9$$

φ M (M, J, G)

[13]

$$L = G^{\bar{\beta}\alpha}\frac{\partial^2}{\partial z^{\alpha}\partial \bar{z}^{\beta}} \quad 10$$

(M_1, g) (M_2, h) m n $M = M_1 \times M_2$ $m + n$ $\pi_1: M \to M_1$ $\pi_2: M \to M_2$ $z = (z_1, z_2) \in M, z_1 = (z^1, \cdots, z^m) \in M_1$ $z_2 = (z^{m+1}, \cdots, z^{m+n}) \in M_2$ $\pi_1(z) = z_1$ $\pi_2(z) = (z_2)$

$d\pi_1: T^{1,0}(M) \to T^{1,0}M_1, d\pi_2: T^{1,0}(M) \to T^{1,0}M_2$ π_1 π_2 $v = (v_1, v_2) \in T_z^{1,0}(M), v_1 = (v^1, \cdots, v^m) \in T_{z_1}^{1,0}M_1, v_2 = (v^{m+1}, \cdots, v^{m+n}) \in T_{z_2}^{1,0}M_2$ $d\pi_1(z,v) = (z_1, v_1)$ $d\pi_2(z,v) = (z_2, v_2)$

[3] (M_1, g) (M_2, h) $f_1: M_1 \to (0, +\infty)$ $f_2: M_2 \to (0, +\infty)$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $G: M \to (0, +\infty)$ $M = M_1 \times M_2$

$$G(z,v) = (f_2 \circ \pi_2)^2(z)g(\pi_1(z), d\pi_1(v)) + (f_1 \circ \pi_1)^2(z)h(\pi_2(z), d\pi_2(v)) \quad 11$$

$z = (z_1, z_2) \in M, v = (v_1, v_2) \in T_z^{1,0}M$ f_1 f_2 (M_1, g) (M_2, h) $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

$f_1 \equiv 1$ $f_2 \equiv 1$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $f_1 \equiv 1$ $f_2 \equiv 1$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ f_1 f_2 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

$1 \leqslant \alpha, \beta, \gamma, \eta, \delta \leqslant m+n, 1 \leqslant i, j, k, l, t \leqslant m, m+1 \leqslant i', j', k', l', t' \leqslant m+n.$ (M_1, g) (M_2, h)

1 2 $\overset{1}{\Gamma}{}^{i}_{jk}$ $\overset{2}{\Gamma}{}^{i'}_{j'k'}$ (M_1, g) (M_2, h)

$({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ (M_1, g) (M_2, h)

$$g_{i\bar{j}} = \frac{\partial^2 g}{\partial v^i \partial \overline{v^j}}, \quad h_{i'\bar{j}'} = \frac{\partial^2 h}{\partial v^{i'} \partial \overline{v^{j'}}} \qquad 12$$

G [3]

$$(G_{\alpha\bar{\beta}}) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right) = \begin{pmatrix} f_2^2 g_{i\bar{j}} & 0 \\ 0 & f_1^2 h_{i'\bar{j}'} \end{pmatrix} \qquad 13$$

$(G^{\bar{\beta}\alpha})$ [3]

$$(G^{\bar{\beta}\alpha}) = \begin{pmatrix} f_2^{-2} g^{\bar{j}i} & 0 \\ 0 & f_1^{-2} h^{\bar{j}'i'} \end{pmatrix} \qquad 14$$

$({}_{f_2}M_1 \times {}_{f_1}M_2, G)$

[3] $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $\Gamma^{\beta}_{\gamma\alpha}$

$$\Gamma^i_{jk} = \overset{1}{\Gamma}{}^i_{jk}, \ \Gamma^i_{j'k} = 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta^i_k, \ \Gamma^{i'}_{jk'} = 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\delta^{i'}_{k'}, \ \Gamma^{i'}_{j'k'} = \overset{2}{\Gamma}{}^{i'}_{j'k'} \qquad 15$$

$$\Gamma^i_{jk'} = \Gamma^i_{j'k'} = \Gamma^{i'}_{jk} = \Gamma^{i'}_{j'k} = 0 \qquad 16$$

$({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $T^{\beta}_{\gamma\alpha}$

$$T^i_{jk} = \overset{1}{T}{}^i_{jk}, \ T^{i'}_{j'k'} = \overset{2}{T}{}^{i'}_{j'k'} \qquad 17$$

$$T^i_{j'k} = 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta^i_k, \quad T^i_{jk'} = -2f_2^{-1}\frac{\partial f_2}{\partial z^{k'}}\delta^i_j \qquad 18$$

$$T^{i'}_{j'k} = -2f_1^{-1}\frac{\partial f_1}{\partial z^k}\delta^{i'}_{j'}, \quad T^{i'}_{jk'} = 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\delta^{i'}_{k'} \qquad 19$$

$$T^i_{j'k'} = T^{i'}_{jk} = 0 \qquad 20$$

2 $\alpha = k, \beta = i, \gamma = j$ 15

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = \overset{1}{\Gamma}{}^i_{jk} - \overset{1}{\Gamma}{}^i_{kj} = \overset{1}{T}{}^i_{jk}$$

$({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ (M_1, g) (M_2, h) f_1 f_2

1 $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ $T^{\beta}_{\gamma\alpha} = 0$ 1

$$\begin{cases} \overset{1}{T}{}^i_{jk} = 0 \\ \overset{2}{T}{}^{i'}_{j'k'} = 0 \\ \dfrac{\partial f_1}{\partial z^k} = 0 \\ \dfrac{\partial f_2}{\partial z^{k'}} = 0 \end{cases} \qquad 21$$

21 (M_1,g) (M_2,h) 21

f_1 f_2

$({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $T_{\gamma\alpha\bar\eta}$

$$T_{jk\bar l} = f_2^2 \overset{1}{T}_{jk\bar l},\ T_{j'k'\bar l'} = f_1^2 \overset{2}{T}_{j'k'\bar l'} \quad (22)$$

$$T_{jk'\bar l} = -2f_2\frac{\partial f_2}{\partial z^{k'}}g_{j\bar l},\ T_{jk'\bar l'} = 2f_1\frac{\partial f_1}{\partial z^j}h_{k'\bar l'} \quad (23)$$

$$T_{j'k\bar l} = 2f_2\frac{\partial f_2}{\partial z^{j'}}g_{k\bar l},\ T_{j'k\bar l'} = -2f_1\frac{\partial f_1}{\partial z^k}h_{j'\bar l'} \quad (24)$$

$$T_{jk\bar l'} = T_{j'k'\bar l} = 0 \quad (25)$$

8 $\alpha = k, \gamma = j, \eta = l$

$$T_{jk\bar l} = G_{\beta\bar l}T^\beta_{jk} = G_{i\bar l}T^i_{jk} + G_{i'\bar l}T^{i'}_{jk} \quad (26)$$

17 13 26

$$T_{jk\bar l} = f_2^2 g_{i\bar l}\overset{1}{T}{}^i_{jk} = f_2^2\overset{1}{T}_{jk\bar l}$$

$({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ τ_γ

$$\tau_j = \overset{1}{\tau}_j + 2f_1^{-1}\frac{\partial f_1}{\partial z^j} \quad (27)$$

$$\tau_{j'} = \overset{2}{\tau}_{j'} + 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}} \quad (28)$$

7 $\gamma = j$

$$\tau_j = G^{\bar\eta\alpha}T_{j\alpha\bar\eta} = G^{\bar lk}T_{jk\bar l} + G^{\bar lk'}T_{jk'\bar l} + G^{\bar l'k}T_{jk\bar l'} + G^{\bar l'k'}T_{jk'\bar l'} \quad (29)$$

14 2 29

$$\begin{aligned}\tau_j &= f_2^{-2}g^{\bar lk}f_2^2\overset{1}{T}_{jk\bar l} + f_1^{-2}h^{\bar l'k'}2f_1\frac{\partial f_1}{\partial z^j}h_{k'\bar l'}\\ &= g^{\bar lk}\overset{1}{T}_{jk\bar l} + 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\\ &= \overset{1}{\tau}_j + 2f_1^{-1}\frac{\partial f_1}{\partial z^j}\end{aligned}$$

28

$({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$

$$\begin{cases}\overset{1}{\tau}_j + 2f_1^{-1}\dfrac{\partial f_1}{\partial z^j} = 0\\ \overset{2}{\tau}_{j'} + 2f_2^{-1}\dfrac{\partial f_2}{\partial z^{j'}} = 0\end{cases} \quad (30)$$

2 $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ $\tau_\gamma = 0$

$$\begin{cases}\tau_j = 0\\ \tau_{j'} = 0\end{cases} \quad (31)$$

27 28 31 30

$({}_{f_2}M_1 \times {}_{f_1}M_2,G)$ f_1 f_2 $({}_{f_2}M_1 \times {}_{f_1}M_2,G)$

(M_1,g) (M_2,h)

[3] (

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